

数理統計学 練習問題 1 [解答]

1. (i) 定義により  $P(B|A) = \frac{P(A \cap B)}{P(A)}$ ,  $P(C|A \cap B) = \frac{P(A \cap B \cap C)}{P(A \cap B)}$

$$(右辺) = P(A) \cdot \frac{P(A \cap B)}{P(A)} \cdot \frac{P(A \cap B \cap C)}{P(A \cap B)} = P(A \cap B \cap C) = (左辺)$$

(ii) 人が嘘をつく事象を  $A_1$ , 真実を述べる事象を  $A_2$ , 嘘発見器が嘘であると判定する事象を  $B$  とおく.

問題文により  $P(B|A_1) = 0.9$ ,  $P(B|A_2) = 0.05$ . また, この人が嘘をつくのか真実を言うのかが全くわからないというので  $P(A_1) = P(A_2) = 0.5$  と考えてよい. 求めるは  $P(A_1|B)$  である.

ベイズの定理により,

$$P(A_1|B) = \frac{P(A_1)P(B|A_1)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2)} = \frac{0.5 \times 0.9}{0.5 \times 0.9 + 0.5 \times 0.05} = \frac{18}{19}$$

2. (i) 確率変数  $X$  は二項分布  $B(n, p)$  に従うので, 確率変数  $2^X$  の期待値  $E[2^X]$ , 分散  $V[2^X]$  は以下の計算で導ける.

$$\begin{aligned} E[2^X] &= \sum_{r=0}^{\infty} 2^r {}_nC_r p^r (1-p)^{n-r} = \sum_{r=0}^{\infty} {}_nC_r (2p)^r (1-p)^{n-r} \\ &= (2p + (1-p))^n = (p+1)^n \end{aligned}$$

$$V[2^X] = E[(2^X)^2] - E[2^X]^2 = E[4^X] - E[2^X]^2 = (3p+1)^n - (p+1)^{2n}$$

(ii)  $M_X(t) = E[e^{tX}] = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda} \lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} = e^{-\lambda} \cdot e^{\lambda e^t} = e^{\lambda(e^t - 1)}$

$$E[X] = M_X'(0) = \lambda e^{(\lambda(e^t - 1) + t)} \Big|_{t=0} = \lambda e^0 = \lambda$$

$$M_X''(0) = \lambda e^{(\lambda(e^t - 1) + t)} \cdot (\lambda e^t + 1) \Big|_{t=0} = \lambda(\lambda + 1)$$

$$V[X] = M_X''(0) - [M_X'(0)]^2 = \lambda(\lambda + 1) - \lambda^2 = \lambda$$

$$\begin{aligned}
3. \quad E[X] &= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}\sigma^2} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx = \int_{-\infty}^{\infty} (\sigma z + \mu) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \quad \left( z = \frac{x-\mu}{\sigma} \text{ と変換する} \right) \\
&= \sigma \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z e^{-\frac{1}{2}z^2} dz + \mu \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} dz \\
&= \sigma \frac{1}{\sqrt{2\pi}} \left[ -e^{-\frac{1}{2}z^2} \right]_{-\infty}^{\infty} + \mu \quad \left( \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} dz = \sqrt{2\pi} \right) \\
&= \mu
\end{aligned}$$

$$\begin{aligned}
V[X] &= \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}\sigma^2} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx - \mu^2 \quad ( V[X] = E[X^2] - E[X]^2 ) \\
&= \int_{-\infty}^{\infty} (\sigma z + \mu)^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz - \mu^2 \\
&= \sigma^2 \int_{-\infty}^{\infty} z^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz + 2\sigma\mu \int_{-\infty}^{\infty} z \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz + \mu^2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz - \mu^2 \\
&= \sigma^2 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-\frac{1}{2}z^2} dz + 0 + \mu^2 - \mu^2 = \sigma^2 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-\frac{1}{2}z^2} dz \\
&= \sigma^2 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z \left( z e^{-\frac{1}{2}z^2} \right) dz \\
&= \sigma^2 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z \left( e^{-\frac{1}{2}z^2} \right) dz - \sigma^2 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left( -e^{-\frac{1}{2}z^2} \right) dz \quad (\text{部分積分}) \\
&= 0 + \sigma^2 = \sigma^2
\end{aligned}$$

4. 「確率変数  $W$  と  $Z$  が独立  $\iff E[WZ] = E[W]E[Z]$  が成り立つ .」  
 という主張があるので,  $E[WZ] - E[W]E[Z] = 0$  となるように  $a$  の値を定めればよい .

$$\begin{aligned}
\text{仮定より } V[X] &= E[X^2] - E[X]^2 = 2, V[Y] = E[Y^2] - E[Y]^2 = 2, \text{cov}(X, Y) = E[XY] - E[X]E[Y] = 1 \\
E[W] &= E[4X + aY] = 4E[X] + aE[Y] \\
E[Z] &= E[2X + Y] = 2E[X] + E[Y] \\
E[W]E[Z] &= 8E[X]^2 + (4+2a)E[X]E[Y] + aE[Y]^2
\end{aligned}$$

$$\begin{aligned}
E[WZ] &= E[(4X + aY)(2X + Y)] = E[8X^2 + (4+2a)XY + aY^2] = 8E[X^2] + (4+2a)E[XY] + aE[Y^2] \\
E[WZ] - E[W]E[Z] &= 8(E[X^2] - E[X]^2) + (4+2a)(E[XY] - E[X]E[Y]) + a(E[Y^2] - E[Y]^2) \\
&= 8 \cdot 2 + (4+2a) \cdot 1 + a \cdot 2 = 16 + (4+2a) + 2a \\
&= 4a + 20
\end{aligned}$$

よって, 求めるは  $a = -5$  .